

ON THE EXISTENCE OF A FACTORIZED UNBOUNDED OPERATOR BETWEEN FRÉCHET SPACES

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ABSTRACT. For locally convex spaces X and Y , the continuous linear map $T : X \rightarrow Y$ is called bounded if there is a zero neighborhood U of X such that $T(U)$ is bounded in Y . Our main result is that the existence of an unbounded operator T between Fréchet spaces E and F which factors through a third Fréchet space G ends up with the fact that the triple (E, G, F) has an infinite dimensional closed common nuclear Köthe subspace, provided that F has the property (y) .

Dedicated to the memory of Prof. Dr. Tosun Terzioğlu

1. INTRODUCTION

Let X and Y be locally convex spaces. A continuous linear map $T : X \rightarrow Y$ is called bounded if there is a θ -neighborhood U of X such that $T(U)$ is bounded in Y . We say that a triple (X, Z, Y) has the bounded factorization property and write $(X, Z, Y) \in \mathcal{BF}$ if each linear continuous operator $T : X \rightarrow Y$ that factors over Y (that is, $T = R_1 R_2$, where $R_2 : X \rightarrow Z$ and $R_1 : Z \rightarrow Y$ are linear continuous operators) is bounded. Nurlu and Terzioğlu [2] proved that under some conditions, existence of continuous linear unbounded operators between nuclear Köthe spaces causes existence of common basic subspaces. Djakov and Ramanujan [1] sharpened this work by removing nuclearity assumption and using a weaker splitting condition. In [6], it is shown that the existence of an unbounded factorized operator for a triple of Köthe spaces, under some assumptions, implies the existence of a common basic subspace for at least two of the spaces. Concerning the class of general Fréchet spaces, the existence of an unbounded operator inbetween is studied in [5]. It is proved that there is an infinite

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dimensional closed common nuclear subspace when the range space has a basis, and admits a continuous norm. When the range space has the property (y), that implies the existence of a common nuclear Köthe quotient as proved in [4]. Combining these two results, when the range space has the property (y), common nuclear Köthe subspace is obtained in [3, Proposition 1]. The aim of this note is to prove the Fréchet space analogue of [6, Proposition 6], that is, under the condition that F has property (y), and $(E, G, F) \notin \mathcal{BF}$ then there is a common nuclear subspace for all three spaces. We rule out the condition where G can be written as $G = \omega \times X$, where X is a Banach space to avoid the case T becomes almost bounded [8].

The locally convex space E with neighborhood base $\mathcal{U}(E)$ is said to have property (y) if there is a neighborhood $U_1 \in \mathcal{U}(E)$ such that

$$E' = \bigcup_{U \in \mathcal{U}(E)} \overline{E'[U_1^\circ] \cap U^\circ}.$$

2. MAIN RESULT

Theorem 2.1. *Let E, F, G be Fréchet spaces where F has property (y). Assume there is a continuous, linear, unbounded operator $T : E \rightarrow F$ which factors through G such as $T = RS$. Then, there exists an infinite dimensional nuclear Köthe subspace M of E such that the restriction $T|_M$ and the restriction $R|_{S(M)}$ are isomorphisms.*

Proof. Let $T : E \rightarrow F$ be an unbounded operator factoring through $G \neq \omega \times X$, for any Banach space X .

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ S \downarrow & \nearrow R & \\ G & & \end{array}$$

By [3, Proposition 1], there exists an infinite dimensional closed nuclear Köthe subspace M of E such that the restriction $T|_M$ is an isomorphism onto $T(M)$. Since T is injective on M , R is injective on $S(M)$ and maps $S(M)$ onto $T(M) = R(S(M))$. Using the fact that T is an isomorphism, it is easy to verify that $R|_{S(M)}$ is one-to-one. Now let $y \in \overline{S(M)}$. So find a sequence $(S(m_n))_{n \in \mathbb{N}}$ in $S(M)$ such that $\lim S(m_n) = y$. R is continuous at y , then $RS(m_n) = T(m_n) = Ry \in \overline{T(M)} = T(M)$, since $T(M)$ is closed. Thus $\lim T(m_n) = T(m) = Ry$ for some $m \in M$. Since T is an isomorphism on M , $\lim T^{-1}T(m_n) = T^{-1}T(m)$, that is, $\lim m_n = m$. S is continuous at m , and that implies $\lim S(m_n) = S(m) = y \in S(M)$. Therefore $S(M)$ is closed. Hence

$R : S(M) \rightarrow R(S(M))$ is an isomorphism by the Open Mapping Theorem. \square

As proved in [7, Lemma 2.1] and [7, Theorem 2.3], property (y), which is assumed to be enjoyed by F can be replaced by being locally closed, or being isomorphic to a closed subspace of a Köthe space. It is shown that these conditions are equivalent to have the property (y).

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